INTEGRABILITY AND CONCENTRATION OF SAMPLE PATHS' TRUNCATED VARIATION OF FRACTIONAL BROWNIAN MOTIONS, DIFFUSIONS AND LÉVY PROCESSES

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ABSTRACT. For a real càdlàg function f defined on a compact interval, its truncated variation at the level $c > 0$ is the infimum of total variations of functions uniformly approximating f with accuracy $c/2$ and (in opposite to total variation) is alway finite. In this paper we discuss exponential integrability and concentration properties of the truncated variation of fractional Brownian motions, diffusions and Lévy processes. We develop a special technique based on chaining approach and using it we prove Gaussian concentration of the truncated variation for certain class of diffusions. We also give sufficient and necessary condition for the existence of exponential moment of order $\alpha > 0$ of truncated variation of Lévy process in terms of its Lévy triplet.

1. INTRODUCTION

Let $X = (X(t))_{t \geq 0}$ be a real valued stochastic process with càdlàg trajectories. In general, the total path variation of X on the compact interval $[a; b] \subset [0; +\infty)$, defined as

$$TV(X, [a; b]) = \sup_n \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sum_{i=1}^{n} |X(t_i) - X(t_{i-1})|,$$

may be (and in many most important cases is) a.s. infinite. However, in the neighborhood of every càdlàg path we may easily find a function with finite total variation.

Let f be a càdlàg function $f : [a; b] \to \mathbb{R}$ and let $c > 0$. The natural question arises, what is the smallest possible (or the greatest lower bound for) total variation of functions from the ball $B(f, c/2) = \{g : \|f - g\|_\infty \leq c/2\}$, where $\|f - g\|_\infty := \sup_{s \in [a;b]} |f(s) - g(s)|$. Some bound from below reads as

$$TV(g, [a; b]) \geq TV^c(f, [a; b]),$$

where
(1.1)

$$TV^c\left(f,[a;b]\right) := \sup_n \sup_{a \le t_0 < t_1 < \ldots < t_n \le b} \sum_{i=1}^n \max\left\{\left|f\left(t_i\right) - f\left(t_{i-1}\right)\right| - c, 0\right\}$$

and follows immediately from the inequality

$$\left|g\left(t_i\right) - g\left(t_{i-1}\right)\right| \ge \max\left\{\left|f\left(t_i\right) - f\left(t_{i-1}\right)\right| - c, 0\right\}.$$

It is possible to show (cf. [9]) that in fact we have equality

(1.2) $\qquad \inf\left\{TV\left(g,[a;b]\right) : \|f - g\|_\infty \le c/2\right\} = TV^c\left(f,[a;b]\right)$

attained for some function f^c from the ball $B(f, c/2)$.

Remark 1. *Since we deal with càdlàg functions, more natural setting of our problem would be the investigation of*

$$\inf\left\{TV\left(g,[a;b]\right) : g \text{ - càdlàg}, d_D(f,g) \le c/2\right\},$$

where d_D denotes Skorohod metric. Since the total variation does not depend on the (continuous and strictly increasing) change of argument and the function f^c minimizing $TV\left(g,[a;b]\right)$ appears to be a càdlàg one, solutions of both problems coincide.

The bound (1.1) is called truncated variation and it is finite for any càdlàg function, since every such function may be uniformly approximated by step functions. Moreover, truncated variation is a continuous and convex function of the parameter $c > 0$ (cf. [9]), and it obviously tends to the total variation as $c \downarrow 0$. For a process with paths with a.s. infinite total variation it may be of interest to assess the rate of this convergence.

This was done so far for continuous semimartingales and it appears (cf. [12]) that for any continuous semimartingale X we have that

$$c \cdot TV^c\left(X,[a;b]\right) \to_{c\downarrow 0} \langle X\rangle_b - \langle X\rangle_a \text{ a.s.,}$$

where $\langle . \rangle$ denotes the quadratic variation of X.

For $t \ge 0$ denote $TV^c\left(X,t\right) = TV^c\left(X,[0;t]\right)$. For X being the unique strong solution of the equation $X_0 = 0, dX_t = \mu\left(X_t\right)dt + \sigma\left(X_t\right)dB_t, t \in [0;S]$, driven by a standard Brownian motion B, with μ and σ satisfying some linear growth conditions, we have even stronger result (cf. [12, Theorem 10]).

$$TV^c\left(X,t\right) - \frac{\langle X\rangle_t}{c} \Rightarrow_{c\downarrow 0} \tilde{B}_{\langle X\rangle_t}/\sqrt{3},$$

where $\tilde{B}$ is a standard Brownian motion independent from B and the convergence "$\Rightarrow$" is understood as the weak functional convergence in $C([0;S], \mathbb{R})$ topology.

For a standard Brownian motion and fixed $S > 0$ this result seems to indicate very strong concentration of its truncated variation on the interval $[0;S]$ around S/c, but it still does not tell anything about the tail probabilities of the functional considered. This observation inclined us to study

the integrability and concentration properties of the truncated variation in greater detail.

Some investigation into this direction was undertaken in [10], were the existence of moment generating function of the truncated variation of Brownian motion with drift on the whole real line was proven, but in this paper we obtain much stronger - Gaussian concentration result.

Another incentive for the study of the magnitude of truncated variation for possibly broad class of processes is the pathwise approach to stochastic integration. In [11] it was shown that it is possible to define the stochastic integral with some correction term as an a.s. limit of pathwise Lebesgue-Stieltjes integrals, when the semimartingale integrator is uniformly approximated on compacts by a finite variation process. Thus the truncated variation gives the magnitude of such integrals, more precisely

$$\inf_{\|X-X^c\|_\infty \le c/2} \sup_{\|Y\|_\infty \le 1} \int_0^S Y_- dX^c = TV^c(X, S).$$

Thus, in this paper we study the magnitude of the truncated variation for Gaussian processes, among them for fractional Brownian motions, and for diffusions. Further we also consider Lévy processes. Our main goal it to describe the tail behavior of $TV^c(X, S)$ assuming that X satisfies some increment condition. We use various techniques depending on the assumption we make. At the beginning we use the chaining concept, i.e. we assume that X satisfies some exponential integrability condition on increments and deduce the exponential integrability of the truncated variation. The chaining approach was first used to study problems of sample boundedness of processes on the general index space [5, 6]. The method was developed to give the full description of classes of processes that are sample bounded under certain integrability condition [1, 2, 3, 8, 15] as well as the small ball probability [4]. For a comprehensive study where many analytical examples are given see [16]. In our study we need some modification of the idea, since we are interested in bounding the supremum of special sums of increments rather then the supremum over increments itself. Therefore we have to invent a special random variable of exponential integrability that bounds the truncated variation.

Our main toy example is the class of fractional Brownian motions, i.e. centered Gaussian processes W_H, $H \in (0, 1)$, starting from 0 and such that $\mathbf{E}|W_H(t) - W_H(s)|^2 = |t - s|^{2H}$. One of the corollaries we get is the following concentration inequality

$$\mathbf{P}(TV^c(W_H, S) \ge c^{\frac{H-1}{H}} S(A_H + B_H u)) \le C_H \exp(-u^{2H}), \text{ for } u \ge 0,$$

where A_H, B_H, C_H are constants, moreover one can set $C_H = 1$ for $H \geq \frac{1}{2}$. By the homogeneity of increments we deduce that $\mathbf{E}TV^c(W_H, S)$ is comparable with $c^{\frac{H-1}{H}}S$ and in this way we prove that
(1.3)
$$\mathbf{P}(TV^c(W_H, S) \geq \mathbf{E}TV^c(W_H, S)(\bar{A}_H + \bar{B}_H u)) \leq \bar{C}_H \exp(-u^{2H}), \text{ for } u \geq 0,$$

for some constants $\bar{A}_H, \bar{B}_H, \bar{C}_H$ (again $\bar{C}_H = 1$ for $H \geq \frac{1}{2}$). In fact any process with similar properties as the fractional Brownian motion, i.e. with homogeneity of increments and exponential integrability of increments can be treated by our method.

Next we turn to investigate the specific fractional Brownian motion, i.e. $W = W_{1/2}$ and diffusions driven by it. Here we can improve our result using the Markov property. It occurs that for Markov processes with moderate growth some local exponential integrability can be extended to the global one. Note that (1.3) implies the existence of the Laplace transform for sufficiently small $\lambda > 0$; assuming the Markov property for diffusions with moderate growth we get the estimate for the Laplace transform on the whole real line. The main result we get in this way is Theorem 2, which for standard Brownian motion implies the following concentration inequality

$$\mathbf{P}(TV^c(W, S) \geq \bar{A}\mathbf{E}TV^c(W, S) + \bar{B}\sqrt{S}u) \leq \exp(-u^2), \text{ for } u \geq 0,$$

where $\bar{A}, \bar{B}$ are universal constants. Therefore the Gaussian concentration inequality holds for the truncated variation of the standard Brownian motion. Our result gives better understanding of the already mentioned phenomenon that $S^{-\frac{1}{2}}(TV^c(W, S) - S/c)$ converges in distribution to $\mathcal{N}(0, 1/3)$ as $c \downarrow 0$.

We continue the study proving sufficient and necessary condition for the finiteness of $\mathbf{E}\exp(\alpha TV^c(X, S))$ for a Lévy process X, in terms of its generating triplet. Here we apply the method of level crossing stopping times.

The structure of the paper is as follows. In Section 2 we introduce the chaining approach which will lead us to the main result on the concentration for processes with increments of exponential decay. Then in Section 3 we discuss the application of the developed methodology to the fractional Brownian motions and then, in Section 4 its improvement for a standard Wiener process and diffusions with moderate growth. In Section 5 we deal with truncated variation of a Lévy process.

Remark 2. *In the whole paper, any dependence of a nonnegative constant on some parameters is always indicated by listing them in brackets or in subscripts, for example, $C(n, \varepsilon)$ or $C_{n,\varepsilon}$.*

2. The chaining approach

In this section we prove fundamental Theorem 1, which will allow us to establish integrability and concentration properties of the truncated variation for a broad class of processes satisfying some increment condition. For simplicity, in this paper we consider processes indexed by a parameter from the metric space (T, d), where T is the compact interval $[0, S]$ equipped with the distance $d(s, t) = |s - t|^q$ for $s, t \in T$, where $0 < q < 1$. Further, let $X(t)$, $t \in T$, be a stochastic process. We assume that there exists $p > 0$ and the function $\varphi_p(x) = 2^{x^p} - 1$ for $x \geq 0$, such that

$$(2.1) \qquad \mathbf{E}\varphi_p \left(\frac{|X(s) - X(t)|}{Cd(s, t)} \right) \leq 1 \ \text{ for } s, t \in T, s \neq t,$$

where $0 \leq C < \infty$ is a universal constant. Condition (2.1) enables us to control the magnitude of the increments of process X, while the truncated variation takes into account only increments greater than c (cf. formula (1.1)). Note that as the consequence of (2.1) and the compactness of T we obtain the existence of a separable modification of $X(t)$, $t \in T$. Then by the linear order of T we can define the càdlàg modification of X which we refer to from now on. Note that the exponential growth of φ_p implies that

$$(2.2) \qquad \varphi_p^{-1}(xy) \leq L_p(\varphi_p^{-1}(x) + \varphi_p^{-1}(y)),$$

for $x, y \geq 0$, where $L_p \in (0; 2]$. Furthermore observe that if $p \geq 1$, $\varphi_p(x)$ is convex on the whole interval $[0; +\infty)$ and if $0 < p < 1$, $\varphi_p(x)$ is convex on the interval $[C_p; +\infty)$ where $C_p = \left(\frac{1-p}{p \ln 2} \right)^{1/p}$. In the same way $\varphi_p(x^q) = \varphi_{pq}(x)$ is convex on the whole interval $[0; +\infty)$ if $pq \geq 1$ and convex on the interval $[C_{p,q}; +\infty)$, where $C_{p,q} = C_{pq}$, if $pq < 1$. We use the notation $C_p, C_{p,q}$ for all $p > 0$, $0 < q < 1$ setting $C_p = 0$ for $p \geq 1$ and $C_{p,q} = 0$ for $pq \geq 1$. Further, we denote $D_p = \varphi_p(C_p)$, $D_{p,q} = \varphi_p(C_{p,q}^q) = \varphi_{pq}(C_{pq})$. Note that $D_{p,q} = 0$ for $pq \geq 1$.

The fundamental result of this paper, from which exponential integrability and concentration properties will follow is

Theorem 1. *Let $X(t)$, $t \in T$, satisfies (2.1) for $\varphi_p = 2^{x^p} - 1$ and $d(s, t) = |s - t|^q$, where where $p > 0$, $0 < q < 1$. Then there exist random variables $Z_1, Z_2 \geq 0$ such that $\mathbf{E}Z_1, \mathbf{E}Z_2 \leq 1$ and for some universal constants $K_1(p, q), K_2(p, q) < \infty$ the following estimate holds*

$$TV^c(X, S) \leq c^{\frac{q-1}{q}} S[K_1(p, q)\varphi_p^{-1}(Z_1 + D_p) + K_2(p, q)[\varphi_p^{-1}]^{\frac{1}{q}}(Z_2 + D_{p,q})].$$

Remark 3. *The main reason why the result holds is that (2.1) gives an exponential decay of increments with large jumps. Therefore we can show a global upper bound on increments in the defined set approximation of the truncated variation. Such an idea is used to bound suprema of processes e.g. [1, 5, 7, 15]. In this paper the main question is to invent common upper bound for an arbitrary sum of truncated increments.*

Remark 4. *The more general setting is possible, namely one may consider exponential like Young functions φ, i.e. positive, increasing, with $\varphi(0) = 0$, $\varphi(1) = 1$ and such that the following condition holds*

$$(2.3) \qquad \varphi^{-1}(xy) \leq L(\varphi^{-1}(x) + \varphi^{-1}(y)), \quad for \ x, y > 0,$$

where $L < \infty$ is a constant. On the other hand we can consider any distance d of the form $d(s,t) = \eta(|s - t|)$, where η is positive, concave, increasing to ∞ and such that $\eta(0) = 0$.

We start with the construction of finite sets approximating T.

2.1. Approximating sequence.

In order to control increments of the process X we approximate space T by a sequence of finite sets $(T_n)_{n=0}^{\infty}, T_n \subset T$. The approximation must be efficient and here we use the entropy of (T, d). Let $N(T, d, \varepsilon)$ be the smallest number of closed balls of radius ε that covers T. More precisely, let us fix $r \geq 4$. We require that T_n is such that $|T_n| = N(T, d, r^{-n}S^q)$ and $\bigcup_{t \in T_n} B_d(t, r^{-n}S) = T$, where $B_d(t, r) = \{s \in T : d(s,t) \leq r\}$. By the linear order of points on the interval it is clear that

$$(2.4) \qquad 2^{-1}r^{\frac{n}{q}} \leq |T_n| < 2^{-1}r^{\frac{n}{q}} + 1.$$

Moreover we may require that $d(s,t) \geq r^{-n}S^q$ for any $s, t \in T_n$, $s \neq t$ and $d(t, T_n) \leq r^{-n}S^q$ for all $t \in T$. Clearly, for any $m = 1, 2, \ldots$

$$(2.5) \qquad \sum_{n=0}^{m} r^{-n}|T_{n+1}| \leq 2^{-1} \sum_{n=0}^{m} r^{-n}(r^{\frac{n+1}{q}} + 2) \leq A(r,q)r^{m\frac{1-q}{q}},$$

where $A(r,q) := r^{\frac{2-q}{q}}(r^{\frac{1-q}{q}} - 1)^{-1}$ (note that $r^{\frac{n+1}{q}} \geq 2$). For each $t \in T_{n+1}$ let $I_{n+1}(t)$ denote the set of $s \in T_{n+1}$ such that $d(s,t) \leq 2r^{-n}S^q$, i.e.

$$(2.6) \qquad I_{n+1}(t) = \left\{s \in T_{n+1} : d(s,t) \leq 2r^{-n}S^q\right\}.$$

Observe that since $|s - t| \geq r^{-\frac{n+1}{q}}S$ for $s, t \in T_{n+1}$, $s \neq t$,

$$(2.7) \qquad |I_{n+1}(t)| \leq \frac{2^{\frac{1}{q}}r^{-\frac{n}{q}}S}{r^{-\frac{n+1}{q}}S} + 1 = 2^{\frac{1}{q}}r^{\frac{1}{q}} + 1 =: B(r,q).$$

Let $\pi_n(t)$ denote the projection of T on T_n, i.e. we require that $\pi_n : T \to T_n$ satisfies $d(t, \pi_n(t)) = d(t, T_n)$. Note that we can define π_n in such a way that it preserves the order on real line i.e. $\pi_n(s) \leq \pi_n(t)$ whenever $s \leq t$. Moreover by the construction $d(t, \pi_n(t)) \leq r^{-n}S^q$ for all $t \in T$.

2.2. Proof of the main theorem.

Our first step is to describe two classes of possible points in a given partition. We fix $\Pi_n = \{t_0, t_1, \ldots, t_n\}$, where $0 \leq t_0 < t_1 < \cdots < t_n \leq S$. Let $J_m = \{i \in N : r^{-m-1}S^q < d(t_{i-1}, t_i) \leq r^{-m}S^q\}$ for $m \geq 0$. The crucial level is $m_0 \geq 0$ such that $r^{-m_0-1}S^q < c \leq$

$r^{-m_0}S^q$. For $i \in J_m$ with $m > m_0$ we will apply exponential concentration. We have

$$\sum_{i=1}^{n}(|X(t_i) - X(t_{i-1})| - c)_+ \le \sum_{m=0}^{m_0}\sum_{i \in J_m}(|X(t_i) - X(t_{i-1})| - c)_+$$

$$(2.8) \quad + \sum_{m=m_0+1}^{\infty}\sum_{i \in J_m}(|X(t_i) - X(t_{i-1})| - c)_+$$

The main tool we will use is the chaining method. We turn to describe the path approximation of t_i for $i \in \{0, 1, \dots, n\}$. Therefore we fix $N \ge 0$ large enough and define $t_i^{N+1} = \pi_{N+1}(t_i)$, then for $l \in \{0, 1, \dots, N\}$ we put by the reverse induction $t_i^l = \pi_l(t_i^{l+1})$. Note that by the construction of π_l we preserve the order of the projections, namely $t_0^l \le t_1^l \le \dots \le t_n^l$ for any $0 \le l \le N + 1$. We require that $N > m_0$ and t_i^N for $i \in \{0, 1, \dots, n\}$ are separated. For a given $i \in J_m$ the level m is the the best to stop the approximation for t_i and move to t_{i-1}. For $i \in J_m$ we split c among all increments. If $m > m_0$ then

$$(|X(t_i) - X(t_{i-1})| - c)_+$$
$$\le (|X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{c}{3})_+ + \sum_{s \in \{i-1,i\}}|X(t_s^{N+1}) - X(t_s)|$$

$$(2.9) \quad + \sum_{l=m+1}^{N}\sum_{s \in \{i-1,i\}}[(|X(t_i^l) - X(t_i^{l+1})| - 2^{-l+m}\frac{c}{3})_+$$

and if $m \le m_0$ then

$$(|X(t_i) - X(t_{i-1})| - c)_+ \le |X(t_i^{m+1}) - X(t_{i-1}^{m+1})|$$

$$+ \sum_{s \in \{i-1,i\}}|X(t_s^{N+1}) - X(t_s)| + \sum_{l=m+1}^{m_0}\sum_{s \in \{i-1,i\}}|X(t_s^l) - X(t_s^{l+1})|$$

$$(2.10) \quad + \sum_{l=m_0+1}^{N}\sum_{s \in \{i-1,i\}}(|X(t_s^l) - X(t_s^{l+1})| - 2^{-l+m_0}\frac{c}{3})_+.$$

Putting together estimates (2.8), (2.9) and (2.10) we obtain the following decomposition lemma.

Lemma 1. *For any partition* $\Pi_n = \{t_0, \dots, t_n\}$, *where* $n \ge 0$, $0 \le t_0 < t_1 < \dots < t_n \le S$ *and* $N > m_0$ *the following estimate holds*

$$\sum_{i=1}^{n}(|X(t_i) - X(t_{i-1})| - c)_+ \le V_1 + V_2 + W_1 + W_2$$

$$+ \sum_{i=1}^{n}\sum_{s \in \{i-1,i\}}|X(t_s) - X(t_s^{N+1})|,$$

where

$$V_1 := \sum_{m=0}^{m_0} \sum_{i \in J_m} \sum_{l=m+1}^{m_0} \sum_{s \in \{i-1,i\}} |X(t_s^l) - X(t_s^{l+1})|;$$

$$W_1 := \sum_{m=0}^{m_0} \sum_{i \in J_m} |X(t_i^{m+1}) - X(t_{i-1}^{m+1})|;$$

$$V_2 := \sum_{m=0}^{m_0} \sum_{i \in J_m} \sum_{l=m_0+1}^{N} \sum_{s \in \{i-1,i\}} (|X(t_s^l) - X(t_s^{l+1})| - 2^{-l+m_0}\frac{c}{3})_+$$

$$+ \sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} \sum_{l=m+1}^{N} \sum_{s \in \{i-1,i\}} (|X(t_s^l) - X(t_s^{l+1})| - 2^{-l+m}\frac{c}{3})_+;$$

$$W_2 := \sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} (|X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{c}{3})_+.$$

In the sequel we will use two simple observations concerning increasing function ψ that is convex starting from some $C_0 \geq 0$, i.e. convex for $x \geq C_0$.

Fact 1. *Let $\psi : [0; +\infty) \to [0; +\infty)$ be a strictly increasing function. Assume that ψ is convex on the interval $[C_0; +\infty)$ where $C_0 \geq 0$, then for any nonnegative $x_1, ..., x_k$ and positive $\alpha_1, ..., \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i \leq M$ we have*

$$(2.11) \qquad \sum_{i=1}^{k} \alpha_i x_i \leq M\psi^{-1}(M^{-1} \sum_{i=1}^{k} \alpha_i \psi(x_i) + \psi(C_0)).$$

Proof. Observe that the function $\bar{\psi}(x) = \psi(x + C_0) - \psi(C_0)$ for $x \geq 0$ is convex, strictly increasing and such that $\bar{\psi}(0) = 0$. Consequently $\bar{\psi}^{-1}(y) = \psi^{-1}(y + \psi(C_0)) - C_0$ is concave with $\bar{\psi}^{-1}(0) = 0$ and using Jensen's inequality we have

$$\sum_{i=1}^{k} \alpha_i x_i \leq \sum_{i=1}^{k} \alpha_i \psi^{-1}(\psi(x_i) + \psi(C_0))$$

$$= \sum_{i=1}^{k} \alpha_i(\bar{\psi}^{-1}(\psi(x_i)) + C_0) \leq MC_0 + M \sum_{i=1}^{k} \frac{\alpha_i}{M} \bar{\psi}^{-1}(\psi(x_i))$$

$$\leq MC_0 + M\bar{\psi}^{-1}\left(\sum_{i=1}^{k} \frac{\alpha_i}{M}\psi(x_i)\right)$$

$$= M\psi^{-1}(M^{-1} \sum_{i=1}^{k} \alpha_i \psi(x_i) + \psi(C_0)).$$

∎

Further, we also have

Fact 2. *For any strictly increasing function $\psi : [0; +\infty) \to [0; +\infty)$ such that ψ is convex on the interval $[C_0; +\infty)$ where $C_0 \geq 0$ and any $M > 0$ we have*

$$(2.12) \qquad \psi^{-1}(y + \psi(C_0)) \leq \max\{M, 1\} \psi^{-1}(y/M + \psi(C_0)).$$

Proof. Again, we consider the function $\bar{\psi}^{-1}$. If $M < 1$, then the thesis follows from the monotonicity of $\bar{\psi}^{-1}$. Now assume that $M \geq 1$. By convexity and $\bar{\psi}^{-1}(0) = 0$, for $y \geq 0$ and $M \geq 1$ we get

$$M\bar{\psi}^{-1}(y/M) \geq \bar{\psi}^{-1}(y),$$

which reads as

$$
\begin{aligned}
M\left(\psi^{-1}(y/M + \psi(C_0)) - C_0\right) &\geq \psi^{-1}(y + \psi(C_0)) - C_0, \\
M\psi^{-1}(y/M + \psi(C_0)) &\geq \psi^{-1}(y + \psi(C_0)) + (M-1)C_0
\end{aligned}
$$

and which gives

$$\psi^{-1}(y + \psi(C_0)) \leq M\psi^{-1}(y/M + \psi(C_0)).$$

$\blacksquare$

Now we turn to estimate the increments for $i \in J_m$, $m \leq m_0$. We have

Lemma 2. *There exists a universal constant $K_1(r, q) < \infty$ and a random variable $Z_1 \geq 0$ independent from the partition Π_n, such that $\mathbf{E}Z_1 \leq 1$ and*

$$V_1 + W_1 \leq K_1(r, q) c^{\frac{q-1}{q}} S \varphi_p^{-1}(Z_1 + D_p).$$

Proof. Recall that $r \geq 4$. Note that for $i \in J_m$, $m \leq N$ the level $m + 1$ is the largest $l \leq N + 1$ such that t_i^l may be equal t_{i-1}^l (i.e. $t_{i-1}^l < t_i^l$ for $l > m + 1$). The reason is that $d(t_i, t_{i-1}) > r^{-m-1}S^q$ and therefore for $l \geq m + 1$

$$d(t_i^{l+1}, t_{i-1}^{l+1}) \geq r^{-m-1}S^q - d(t_{i-1}^{l+1}, t_{i-1}) - d(t_i^{l+1}, t_i)$$

$$\geq r^{-m-1}S^q - 2\sum_{j=l+1}^{\infty} r^{-j}S^q \geq r^{-m-1}S^q - 2\frac{r^{-m-2}S^q}{1 - r^{-1}} > 0.$$

Consequently in the path approximation of Π_n the increment $|X(u) - X(\pi_l(u))|$ with some $u \in T_{l+1}$ can be used at most twice and only for a single t_i from the partition. Moreover, by the entropy construction of T_l we have $d(u, \pi_l(u)) \leq r^{-l}S^q$. It implies that

$$V_1 \leq 2\sum_{l=0}^{m_0} \sum_{u \in T_{l+1}} |X(u) - X(\pi_l(u))|$$

$$(2.13) \qquad \leq \mathcal{V}_1 := 2C\sum_{l=0}^{m_0} r^{-l}S^q \sum_{u \in T_{l+1}} \frac{|X(u) - X(\pi_l(u))|}{Cd(u, \pi_l(u))}.$$

On the other hand $d(t_i, t_{i-1}) \leq r^{-m}S^q$ for $i \in J_m$ and

$$d(t_i^{m+1}, t_{i-1}^{m+1}) \leq d(t_i, t_{i-1}) + d(t_i^{N+1}, t_i) + d(t_{i-1}^{N+1}, t_{i-1})$$

$$+ \sum_{l=m+1}^{N} [d(t_i^{l+1}, t_i^l) + d(t_{i-1}^{l+1}, t_{i-1}^l)] \leq r^{-m}S^q + 2\sum_{l=m+1}^{\infty} r^{-l}S^q \leq 2r^{-m}S^q.$$

Therefore using the defined sets $I_m(u) = \{v \in T_{m+1} : d(u,v) \leq 2r^{-m}S^q\}$, we obtain that

$$W_1 \leq \sum_{l=0}^{m_0} \sum_{u \in T_{l+1}} \sum_{v \in I_{l+1}(u)} |X(u) - X(v)|$$

(2.14) $$\leq \mathcal{W}_1 := C \sum_{l=0}^{m_0} 2r^{-l}S^q \sum_{u \in T_{l+1}} \sum_{v \in I_{l+1}(u)} \frac{|X(u) - X(v)|}{Cd(u,v)}.$$

We calculate the sum of all weights appearing in (2.13) and (2.14). By (2.7) for each $u \in T_{m+1}$ we have $|I_{l+1}(u)| \leq B(r,q)$ and hence, using also (2.5)

$$M_1 := \sum_{l=0}^{m_0} r^{-l}S^q[|T_{l+1}| + \sum_{u \in T_{l+1}} |I_{l+1}(u)|]$$

$$\leq [1 + B(r,q)]S^q \sum_{l=0}^{m_0} r^{-l}|T_{l+1}| \leq A(r,q)[1 + B(r,q)]r^{m_0 \frac{1-q}{q}}S^q.$$

Therefore by $c \leq r^{-m_0}S^q$ we get $r^{m_0 \frac{1-q}{q}}S^q \leq c^{-\frac{1-q}{q}}S$ and hence $M_1 \leq A(r,q)[1 + B(r,q)]c^{\frac{q-1}{q}}S$. Using Fact 1 for φ_p which is convex above C_p we get

(2.15)
$$\mathcal{V}_1 + \mathcal{W}_1 \leq 2CM_1\varphi_p^{-1}(Z_1 + \varphi_p(C_p)) \leq K_1(r,q)c^{\frac{q-1}{q}}S\varphi_p^{-1}(Z_1 + \varphi_p(C_p)),$$

where $K_1(r,q) := 2A(r,q)[1 + B(r,q)]C$ and

$$Z_1 = M_1^{-1} \sum_{l=0}^{m_0} r^{-l}S^q \sum_{u \in T_{l+1}} [\varphi_p(\frac{|X(u) - X(\pi_l(u))|}{Cd(u, \pi_l(u))})$$

$$+ \sum_{v \in I_{l+1}(u)} \varphi_p(\frac{|X(u) - X(v)|}{Cd(u,v)})].$$

Obviously $Z_1 \geq 0$ and $\mathbf{E}Z_1 \leq 1$ by (2.1). Combining (2.13),(2.14) and (2.15) we get the result.   ∎

Our second goal is to prove a bound for increments above the level m_0.

Lemma 3. *There exists a universal constant $K_2(p,q,r) < \infty$ and a random variable $Z_2 \geq 0$ independent from the partition Π_n such that $\mathbf{E}Z_2 \leq 1$ and the following inequality holds*

$$V_2 + W_2 \leq K_2(p,q,r)[\varphi_p^{-1}]^{\frac{1}{q}}(Z_2 + D_{p,q}).$$

Proof. First we prove a bound for V_2. Our main tool is (2.2), which implies that

$$(2.16) \qquad \varphi_p([x-y]_+) \leq \frac{\varphi_p(L_p x)}{\varphi_p(y)}, \quad x, y \geq 0.$$

We analyze the increment

$$(|X(t_i^{l+1}) - X(t_i^l)| - 2^{m-l}\frac{c}{3})_+, \quad l > m, \ i \in J_m, \ m > m_0.$$

Using (2.16) and $d(t_i^{l+1}, t_i^l) \leq r^{-l}S^q$ we obtain that

$$|X(t_i^{l+1}) - X(t_i^l)| - 2^{-l+m}\frac{c}{3} \leq \left[r^{-l}S^q\frac{|X(t_i^{l+1}) - X(t_i^l)|}{d(t_i^{l+1}, t_i^l)} - 2^{-l+m}\frac{c}{6} \right]_+ - 2^{-l+m}\frac{c}{6}$$

$$= CL_p r^{-l}S^q\varphi_p^{-1}(\varphi_p\left(\left[\frac{|X(t_i^{l+1}) - X(t_i^l)|}{CL_p d(t_i^{l+1}, t_i^l)} - \frac{2^{-l+m}r^l c}{6CL_p S^q}\right]_+\right)) - 2^{-l+m}\frac{c}{6}$$

$$\leq CL_p r^{-l}S^q\varphi_p^{-1}(a_{l,m}^{-1}\varphi_p(\frac{|X(t_i^{l+1}) - X(t_i^l)|}{Cd(t_i^{l+1}, t_i^l)})) - b_{l,m}c,$$

where

$$a_{l,m} = \varphi_p(\frac{2^{-l+m}r^l c}{6CL_p S^q}), \quad b_{l,m} = \frac{1}{6}2^{-l+m} \text{ for } l > m.$$

Then we apply $[z-1]_+ \leq z^{1/q}$ for $z \geq 0$ and obtain

$$|X(t_i^{l+1}) - X(t_i^l)| - 2^{-l+m}\frac{c}{3}$$

$$\leq b_{l,m}c\left[(b_{l,m}c)^{-1}CL_p r^{-l}S^q\varphi_p^{-1}(a_{l,m}^{-1}\varphi_p(\frac{|X(t_i^{l+1}) - X(t_i^l)|}{Cd(t_i^{l+1}, t_i^l)})) - 1\right]_+$$

$$\leq (b_{l,m}c)^{\frac{q-1}{q}}(CL_p r^{-l})^{\frac{1}{q}}S[\varphi_p^{-1}]^{\frac{1}{q}}(a_{l,m}^{-1}\varphi_p(\frac{|X(t_i^{l+1}) - X(t_i^l)|}{Cd(t_i^{l+1}, t_i^l)}))$$

$$(2.17) \quad = B_{l,m}c^{\frac{q-1}{q}}S[\varphi_p^{-1}]^{\frac{1}{q}}(a_{l,m}^{-1}\varphi_p(\frac{|X(t_i^{l+1}) - X(t_i^l)|}{Cd(t_i^{l+1}, t_i^l)})),$$

where $B_{l,m} = b_{l,m}^{\frac{q-1}{q}}(CL_p r^{-l})^{\frac{1}{q}}$. Note that similar estimate holds for $|X(t_{i-1}^{l+1}) - X(t_{i-1}^l)| - 2^{-l+m}c/3$.

Now observe that for $i \in J_m$, $m > m_0$, using that $|t_i - t_{i-1}| \geq r^{-\frac{m+1}{q}}S$ we get
(2.18)

$$\sum_{l=m+1}^{N} B_{l,m}S = \sum_{l=m+1}^{N} b_{l,m}^{\frac{q-1}{q}}(CL_p r^{-l})^{\frac{1}{q}}S \leq M_2 r^{-\frac{m+1}{q}}S \leq M_2|t_i - t_{i-1}|,$$

where $M_2 = M_2(r, p, q)$ is defined by

$$\sum_{l=m+1}^{N} b_{l,m}^{\frac{q-1}{q}} (CL_p r^{-l+m+1})^{\frac{1}{q}} = \sum_{l=m+1}^{N} (\frac{1}{6} 2^{-l+m})^{\frac{q-1}{q}} (CL_p r^{-l+m+1})^{\frac{1}{q}}$$

$$\leq (12^{1-q} CL_p)^{\frac{1}{q}} \sum_{l'=0}^{\infty} (2^{\frac{1-q}{q}} r^{-\frac{1}{q}})^{l'} = (12^{1-q} CL_p)^{\frac{1}{q}} (1 - 2^{\frac{1-q}{q}} r^{-\frac{1}{q}})^{-1} =: M_2.$$

If $i \in J_m$, $m \leq m_0$, $l > m_0$ then the same argument works and we get

$$|X(t_i^{l+1}) - X(t_i^l)| - 2^{-l+m_0} \frac{c}{3}$$

$$(2.19) \qquad \leq \bar{B}_l c^{\frac{q-1}{q}} S[\varphi_p^{-1}]^{\frac{1}{q}} (\bar{a}_l^{-1} \varphi_p(\frac{|X(t_i^{l+1}) - X(t_i^l)|}{Cd(t_i^{l+1}, t_i^l)})),$$

where

$$\bar{a}_l = \varphi_p(\frac{2^{-l+m_0} r^l c}{6 CL_p S^q}), \quad \bar{b}_l = \frac{1}{6} 2^{-l+m_0}, \quad \bar{B}_l = \bar{b}_l^{\frac{q-1}{q}} (CL_p r^{-l})^{\frac{1}{q}} \text{ for } l > m_0.$$

Moreover in this case $|t_i - t_{i-1}| \geq r^{-\frac{m_0+1}{q}} S$ and hence again

$$(2.20) \qquad \sum_{l=m_0+1}^{N} \bar{B}_l S \leq M_2 r^{-\frac{m_0+1}{q}} S \leq M_2 |t_i - t_{i-1}|.$$

Using (2.17) and (2.19) we estimate V_2. Denoting

$$\sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} \sum_{l=m+1}^{N} \sum_{s \in \{i-1, i\}} =: \sum_{1}$$

and

$$\sum_{m=0}^{m_0} \sum_{i \in J_m} \sum_{l=m_0+1}^{N} \sum_{s \in \{i-1, i\}} =: \sum_{2}$$

we have

$$V_2 \leq c^{\frac{q-1}{q}} S \sum_{1} B_{l,m}[\varphi_p^{-1}]^{\frac{1}{q}} (a_{l,m}^{-1} \varphi_p(\frac{|X(t_s^{l+1}) - X(t_s^l)|}{Cd(t_i^{l+1}, t_i^l)}))$$

$$+ c^{\frac{q-1}{q}} S \sum_{2} \bar{B}_l [\varphi_p^{-1}]^{\frac{1}{q}} (\bar{a}_l^{-1} \varphi_p(\frac{|X(t_s^{l+1}) - X(t_s^l)|}{Cd(t_i^{l+1}, t_i^l)})).$$

Using (2.18) and (2.20) we estimate the sum of weights appearing in the expression estimating V_2

$$\sum_{1} B_{l,m} + \sum_{2} \bar{B}_l = S^{-1} \sum_{1} B_{l,m} S + S^{-1} \sum_{2} \bar{B}_l S$$

$$\leq S^{-1} 2 M_2 \sum_{i=1}^{n} |t_i - t_{i-1}| + S^{-1} 2 M_2 \sum_{i=1}^{n} |t_i - t_{i-1}| \leq 4 M_2.$$

Using Fact 1 for $\varphi_p(x^q)$ which is convex above $C_{p,q}$ we get

$$(2.21) \qquad V_2 \leq 4M_2 c^{\frac{q-1}{q}} S[\varphi_p^{-1}]^{\frac{1}{q}} (\bar{V}_2 + D_{p,q}),$$

where

$$\bar{V}_2 = (4M_2)^{-1} \sum_1 B_{l,m} a_{l,m}^{-1} \varphi_p \left(\frac{|X(t_s^{l+1}) - X(t_s^l)|}{Cd(t_s^{l+1}, t_s^l)} \right)$$
$$+ (4M_2)^{-1} \sum_2 \bar{B}_l \bar{a}_l^{-1} \varphi_p \left(\frac{|X(t_s^{l+1}) - X(t_s^l)|}{Cd(t_s^{l+1}, t_s^l)} \right).$$

Now note that for $m \geq m_0$

$$B_{l,m} a_{l,m}^{-1} \leq B_{l,m_0} a_{l,m_0}^{-1} = \bar{B}_l \bar{a}_l^{-1}.$$

Moreover again we use the fact that for the path approximation of Π_n and given $u \in T_{l+1}$ the increment $X(u) - X(\pi_l(u))$ can be used only twice and for a single t_i from the partition. It implies that

$$(2.22) \quad \bar{V}_2 \leq \mathcal{V}_2 := (4M_2)^{-1} \sum_{l=m_0+1}^{\infty} \bar{B}_l \bar{a}_l^{-1} \sum_{u \in T_{l+1}} \varphi \left(\frac{|X(u) - X(\pi_l(u))|}{Cd(u, \pi_l(u))} \right).$$

Observe that by (2.1) and (2.4)

$$\mathbf{E}\mathcal{V}_2 \leq (4M_2)^{-1} \sum_{l=m_0+1}^{\infty} \bar{B}_l \bar{a}_l^{-1} |T_{l+1}|$$

$$\leq (4M_2)^{-1} \sum_{l=m_0+1}^{\infty} \bar{a}_l^{-1} \bar{b}_l^{\frac{q-1}{q}} (CL_p r^{-l})^{\frac{1}{q}} 2^{-1} (r^{\frac{l+1}{q}} + 1)$$

$$\leq (4M_2)^{-1} (6^{1-q} CL_p)^{\frac{1}{q}} \sum_{l=m_0+1}^{\infty} 2^{(-l+m_0)\frac{q-1}{q}} \varphi_p \left(\frac{2^{-l+m_0} r^l c}{6CL_p S^q} \right)^{-1}.$$

Therefore due to $r^{m_0+1} c/S^q > 1$ we obtain that

$$\mathbf{E}\mathcal{V}_2 \leq (4M_2)^{-1} (6^{1-q} CL_p r)^{\frac{1}{q}} \sum_{l'=1}^{\infty} 2^{l' \frac{1-q}{q}} \varphi_p \left(\frac{2^{-l'} r^{m_0+1+l'} c}{6CL_p S^q r} \right)^{-1}$$

$$\leq (4M_2)^{-1} (6^{1-q} CL_p r)^{\frac{1}{q}} \sum_{l'=1}^{\infty} 2^{l' \frac{1-q}{q}} \varphi_p \left(\frac{(r/2)^{l'}}{6CL_p r} \right)^{-1} := M_3.$$

Finally, by (2.21), (2.22) and Fact 2, we get

$$(2.23) \qquad V_2 \leq 4M_2 \max\{M_3, 1\} c^{\frac{q-1}{q}} S[\varphi_p^{-1}]^{\frac{1}{q}} (\mathcal{V}_2/M_3 + D_{p,q}).$$

Note that $\mathbf{E}\mathcal{V}_2/M_3 \leq 1$.

A similar argument can be used to bound increments in W_2, namely for $m > m_0$

$$|X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{1}{3}c$$

$$\leq CL_p r^{-m-1} S^q \varphi_p^{-1}(\varphi_p([\frac{|X(t_i^{m+1}) - X(t_{i-1}^{m+1})|}{L_p Cd(t_i^{m+1}, t_{i-1}^{m+1})} - \frac{r^{m+1}c}{6CL_p S^q}]_+)) - \frac{c}{6}$$

$$\leq CL_p r^{-m-1} S^q \varphi_p^{-1}(\bar{a}_m^{-1} \varphi_p(\frac{|X(t_i^{m+1}) - X(t_{i-1}^{m+1})|}{Cd(t_i^{m+1}, t_{i-1}^{m+1})})) - \frac{c}{6},$$

where

$$\hat{a}_m = \varphi_p(\frac{r^{m+1}c}{6CL_p S^q}) \text{ for } m > m_0.$$

Then by $[z-1]_+ \leq z^{\frac{1}{q}}$ for $z \geq 0$

$$|X(t_i^{m+1}) - X(t_{i-1}^{m+1})| - \frac{c}{3}$$

$$\leq \frac{c}{6} \left[6c^{-1} CL_p r^{-m-1} S^q \varphi_p^{-1}(\hat{a}_m^{-1} \varphi_p(\frac{|X(t_i^{m+1}) - X(t_{i-1}^{m+1})|}{Cd(t_i^{m+1}, t_{i-1}^{m+1})})) - 1 \right]_+$$

$$\leq \left(\frac{c}{6}\right)^{\frac{q-1}{q}} (CL_p r^{-m-1})^{\frac{1}{q}} S[\varphi_p^{-1}]^{\frac{1}{q}} (\hat{a}_m^{-1} \varphi_p(\frac{|X(t_i^{m+1}) - X(t_{i-1}^{m+1})|}{Cd(t_i^{m+1}, t_{i-1}^{m+1})})).$$

Let $M_4 := (6^{1-q} CL_p)^{\frac{1}{q}}$. Using that $r^{-\frac{m+1}{q}} S \leq |t_i - t_{i-1}| \leq r^{-\frac{m}{q}} S$ we get

$$\sum_{m=m_0+1}^{\infty} \sum_{i \in J_m} (6^{1-q} CL_p r^{-m-1})^{\frac{1}{q}} S \leq M_4 \sum_{i=1}^{n} |t_i - t_{i-1}| = M_4 S.$$

Therefore using Fact 1 for $\varphi_p(x^q)$ we get

(2.24) $$W_2 \leq M_4 c^{\frac{q-1}{q}} S[\varphi_p^{-1}]^{\frac{1}{q}} (\bar{W}_2 + D_{p,q}),$$

where

$$\bar{W}_2 := M_4^{-1} \sum_{m=m_0+1}^{N} \sum_{i \in J_m} \hat{a}_m^{-1} (6^{1-q} CL_p r^{-m-1})^{\frac{1}{q}} \varphi_p(\frac{|X(t_i^{m+1}) - X(t_{i-1}^{m+1})|}{Cd(t_i^{m+1}, t_{i-1}^{m+1})}).$$

Clearly

$$\bar{W}_2 \leq \mathcal{W}_2$$

$$:= M_4^{-1} \sum_{m=m_0+1}^{\infty} \hat{a}_m^{-1} (6^{1-q} CL_p r^{-m-1})^{\frac{1}{q}} \sum_{u \in T_{m+1}} \sum_{v \in I_{m+1}(u)} \varphi_p(\frac{|X(u) - X(v)|}{Cd(u,v)}).$$

Note that by (2.4), (2.7)

$$\sum_{u \in T_{m+1}} |I_{m+1}(u)| \leq 2^{-1} B(r,q)(r^{\frac{m+1}{q}} + 1) \leq B(r,q) r^{\frac{m+1}{q}}.$$

Moreover $r^{m_0+1}c/S^q > 1$ and hence

$$\mathbf{E}\mathcal{W}_2 = M_4^{-1}B(r,q)\sum_{m=m_0+1}^{\infty}\hat{a}_m^{-1}(6^{1-q}CL_pr^{-m-1})^{\frac{1}{q}}r^{\frac{m+1}{q}}$$

$$\leq M_4^{-1}(6^{1-q}CL_p)^{\frac{1}{q}}B(r,q)\sum_{m=m_0+1}^{\infty}\varphi_p(\frac{r^{m+1}c}{6CL_pS^q})^{-1}$$

$$\leq B(r,q)\sum_{m'=1}^{\infty}\varphi_p\left(\frac{r^{m'}}{6CL_p}\right)^{-1} =: M_5.$$

By (2.24) and Fact 2 we get

(2.25) $\qquad W_2 \leq M_4\max\{M_5,1\}c^{\frac{q-1}{q}}S[\varphi_p^{-1}]^{\frac{1}{q}}(\mathcal{W}_2/M_5 + D_{p,q}).$

Since $\mathbf{E}\mathcal{W}_2/M_5 \leq 1$ by (2.23), (2.25) and Jensen's inequality we obtain the thesis.

∎

Now we are ready to finish the proof of Theorem 1.
Proof.(of Theorem 1) It suffices to use Lemma 1, then universal bounds given in Lemmas 2, 3 and finally let $N \to \infty$. Recall that by the construction $\lim_{N\to\infty} d(t,\pi_{N+1}(t)) = 0$ for any $t \in T$. Note that we can optimize the inequalities with respect to $r \geq 4$ and therefore constants K_1 and K_2 depend on p and q only.

∎

The meaning of the result the is that for $p > 0$ and $0 < q < 1$ there holds some concentration inequality. To formulate the results in the elegant way observe that there exists $E_q \in [0;1]$ such that $E_q + x^{\frac{1}{q}} \geq x$ for $x \geq 0$ and hence, due to $D_{p,q} \leq D_p$, we get

(2.26) $\qquad E_q + [\varphi_p^{-1}]^{\frac{1}{q}}(x + D_{p,q}) \geq \varphi_p^{-1}(x + D_p)$ for $x \geq 0.$

As the consequence of (2.26) and Jensen's inequality we get

Corollary 1. *Under the assumptions of Theorem 1 there exist r.v. Z such that $Z \geq 0$, $\mathbf{E}Z \leq 1$ and for some constants $A(p,q)$, $B(p,q)$ the following estimate holds*

$$TV^c(X,S) \leq c^{\frac{q-1}{q}}S[A(p,q) + B(p,q)[\varphi_p^{-1}]^{\frac{1}{q}}(Z + D_{p,q})].$$

Application of Chebyshev's inequality immediately gives

Corollary 2. *The following inequality holds*

$$\mathbf{P}\left(TV^c(X,S) \geq c^{\frac{q-1}{q}}S[\bar{A}(p,q) + \bar{B}(p,q)u]\right) \leq \bar{D}_{p,q}\exp(-u^{pq}), \quad for\ u > 0.$$

where $\bar{A}(p,q), \bar{B}(p,q)$ are universal constants, e.g. $\bar{A}(p,q) = A(p,q)+(2/\ln 2)^{\frac{1}{pq}}B(p,q)$, $\bar{B}(p,q) = (2/\ln 2)^{\frac{1}{pq}}B(p,q)$ and $\bar{D}_{p,q} = D_{p,q} + 1$. In particular $\bar{D}_{p,q} = 1$ for $pq \geq 1$.

3. Application to the fractional Brownian motion

Now we consider $T = [0, S]$ with distance $d(s, t) = |t - s|^H$, where $H \in (0, 1)$. Let $W_H(t)$, $t \in T$, be a fractional Brownian motion with the Hurst coefficient H. Then for some constant $C(H)$

$$\mathbf{E}\varphi_2\left(\frac{|W_H(t) - W_H(s)|}{C(H)|t - s|^H}\right) \leq 1, \quad \text{for } s, t \in T.$$

Consequently all the assumption of Theorem 1 are satisfied and we get

Corollary 3. *For any fractional Brownian motion $W_H(t)$, $t \in T$, the following inequality holds*

$$\mathbf{P}\left(TV^c(W_H, S) \geq c^{\frac{H-1}{H}} S(A_H + B_H u)\right) \leq C_H \exp(-u^{2H}), \quad \text{for } u > 0.$$

where A_H, B_H, C_H are universal constants and $C_H = 1$ for $H \geq 1/2$.

Note that Corollary 3 implies that $\mathbf{E}TV^c(W_H, S) \leq K_H c^{\frac{H-1}{H}} S$. On the other hand $c^{\frac{H-1}{H}} S$ is also the proper lower bound for $\mathbf{E}TV^c(X, S)$. Indeed, let us consider the partition Π given by the entropy $N(T, d, c)$, i.e. Π such that $|\Pi| = N(T, d, c)$ and $\bigcup_{t \in \Pi} B_d(t, c) = T$. In particular the construction gives that $d(s, t) \geq c$ for $s \neq t$, $s, t \in \Pi$. Thus denoting $\Pi = \{t_1, ..., t_N\}$ and $0 = t_0 < t_1 < ... < t_N < S$ we have

$$TV^c(W_H, S) \geq \sum_{i=1}^{N} (|X(t_i) - X(t_{i-1})| - c)_+.$$

Clearly $N \sim c^{-\frac{1}{H}} S$ and $\mathbf{E}(|W_H(t_i) - W_H(t_{i-1})| - c)_+ \sim c$ due to $c \leq d(t_i, t_{i-1}) \leq 2^H c$. It proves that $c^{\frac{H-1}{H}} S$ is comparable with $\mathbf{E}TV^c(W_H, S)$ up to a constant depending only on H. Therefore we have another formulation of Corollary 1.

Corollary 4. *For any fractional Brownian motion $W_H(t)$, $t \in T$, the following inequality holds*

$$\mathbf{P}\left(TV^c(W_H, S) \geq \mathbf{E}TV^c(W_H, S)(\bar{A}_H + \bar{B}_H u)\right) \leq \bar{C}_H \exp(-u^{2H}), \quad \text{for } u > 0,$$

where $\bar{A}_H, \bar{B}_H, \bar{C}_H$ are constants. Moreover $\bar{C}_H = 1$, for $H \geq 1/2$.

4. Application to the standard Brownian motion and diffusions

For a standard Brownian motion $W = W_{1/2}$, which is the only fractional Brownian motion with independent increments one may, using this property, strengthen the results obtained for general fBm and obtain Gaussian concentration of $TV^c(W, S)$. Generalization for diffusions driven by W, with moderate growth is also possible.

Let us assume that $X_t, t \geq 0$, is a one-dimensional diffusion satisfying

$$(4.1) \qquad X(t) = x_0 + \int_0^t \mu(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s)$$

We assume that $\sigma : [0; +\infty) \times \mathbb{R} \to [-R; R]$ is measurable and bounded (i.e. $0 < R < +\infty$) and $\mu : [0; +\infty) \times \mathbb{R} \to \mathbb{R}$ is measurable and satisfying the following linear growth condition: there exists $C, D \geq 0$ such that for all $t \geq 0$

$$(4.2) \qquad |\mu(t, x)| \leq C + D|x|.$$

We will also need the natural assumption that X is a Markov process. With this assumption we have

Theorem 2. *For X being a Markov process satisfying (4.1) with μ and σ as above and $\lambda \geq 0$ one has*

$$\mathbf{E}\exp\left(\lambda TV^c(X, S)\right) \leq 2\exp\left(\lambda^2 S\alpha_R + \lambda S c^{-1}\beta_R + \lambda\gamma_{x_0, C, D, S}\right) \times$$
$$\times \left(1 + 8\lambda\eta_{D, R, S}\exp\left(\lambda^2\eta_{D, R, S}^2\right)\right),$$

where $\gamma_{x_0, C, D, S} = (C + D|x_0|)Se^{DS}$, $\delta_{D, S} = DSe^{DS}$ and $\eta_{D, R, S} = \delta_{D, S}R\sqrt{S/2}$. In particular, for $D = 0$ we get

$$\mathbf{E}\exp\left(\lambda TV^c(X, S)\right) \leq 2\exp\left(\lambda^2 S\alpha_R + \lambda S\left(c^{-1}\beta_R + C\right)\right)$$

and for the standard Brownian motion $X = W$ we get

$$(4.3) \qquad \mathbf{E}\exp\left(\lambda TV^c(W, S)\right) \leq 2\exp\left(\lambda^2 S\alpha + \lambda S c^{-1}\beta\right),$$

where α, β are universal constants.

Proof. We have Let us define

$$M(t) := \int_0^t \mu(s, X(s))\,ds, \quad Y(t) := \int_0^t \sigma(s, X(s))\,dW(s)$$

and $Y^* = \sup_{0 \leq s \leq S}|Y(s)|$. We have $X(t) = x_0 + M(t) + Y(t)$, and due to (4.2) we estimate

$$|M(t)| \leq \int_0^t |\mu(s, X(s))|\,ds \leq \int_0^t C + D|X(s)|\,ds$$
$$\leq \int_0^t C + D|x_0| + D|M(s)| + DY^*\,ds$$
$$(4.4) \qquad \leq (C + D|x_0| + DY^*)S + D\int_0^t |M(s)|\,ds.$$

Hence, from Gronwall's lemma we get

$$(4.5) \qquad |M(t)| \leq (C + D|x_0| + DY^*)Se^{Dt}.$$

Notice that due to (4.5) M is adapted, absolute continuous process with locally bounded total variation. Indeed, repeating estimates (4.4) and using

(4.5) we get

$$TV(M,S) \leq \int_0^S |\mu(s, X(s))| \, ds \leq (C + D|x_0| + DY^*) S + D \int_0^S |M(t)| \, ds$$

$$\leq (C + D|x_0| + DY^*) S + D(C + D|x_0| + DY^*) S \int_0^S e^{Dt} ds$$

$$(4.6) \quad = (C + D|x_0|) Se^{DS} + DSe^{DS} Y^*$$

($TV = TV^0$ denotes here total variation.) By [12, Fact 17] we have

$$(4.7) \qquad\qquad TV^c(X, S) \leq TV(M, S) + TV^c(Y, S).$$

Now we will investigate $TV^c(Y, S)$. First, let us prove that Y satisfies condition (2.1) with $p = 2$ and $d(s,t) = |s - t|^{1/2}$. Indeed, let us fix $0 \leq s < t \leq S$ and consider the following martingale $Z(u) := Y(s + u) - Y(s)$, $u \in [0; t - s]$. We have

$$Z(u) = \int_s^{s+u} \sigma(\tau, X(\tau)) \, dW(\tau)$$

and

$$\langle Z \rangle (u) = \int_s^{s+u} \sigma(\tau, X(\tau))^2 \, d\tau \leq R^2 (t - s).$$

Hence, by Bernstein's inequality (cf. [13, Chapt. IV, Exercise 3.16]), we have

$$\mathbf{P}(|Y(t) - Y(s)| \geq x) \leq 2\mathbf{P}\left(\sup_{u \in [0; t-s]} Z(u) \geq x \right)$$

$$= 2\mathbf{P}\left(\sup_{u \in [0; t-s]} Z(u) \geq x, \langle Z \rangle (t - s) \leq R^2 (t - s) \right)$$

$$(4.8) \qquad \leq 2\exp\left(-x^2 / \left(2R^2 (t - s) \right) \right).$$

From (4.8) we immediately get that Y satisfies condition (2.1) for $p = 2$ and $d(s,t) = |s - t|^{1/2}$. Hence, from Corollary 2 we obtain the following bound on the tails of $TV^c(Y, S)$:

$$(4.9) \qquad\qquad \mathbf{P}\left(TV^c(Y, S) \geq c^{-1} S(A + Bu) \right) \leq e^{-u},$$

where $A = A(R)$ and $B = B(R)$ depend on R only. Notice that for $\delta > 0$ applying Bernstein's inequality to Y^* we have $\mathbf{P}(Y^* \geq x) \leq 2\exp\left(-x^2 / (2R^2 S) \right)$ and using the integration by parts formula we have

$$(4.10) \quad \mathbf{E}\exp(\delta Y^*) \leq 1 + 2\delta \int_0^\infty e^{\delta y} e^{-y^2 / (2R^2 S)} dy \leq 1 + 8\delta R \sqrt{S/2} e^{\delta^2 R^2 S/2}.$$

Now, we will strengthen estimate (4.9) using the Markov property of X. First, using (4.9) and integration by parts we have

$$(4.11) \qquad\qquad \mathbf{E}\exp\left(\lambda \left[TV^c(Y, S) - c^{-1} SA \right] \right) \leq \frac{1}{1 - \lambda SB/c}$$

for $\lambda < c\,(SB)^{-1}$. Let now $S = S_1 + S_2$, where $S_1, S_2 > 0$. Using the inequality $TV^c\,(Y, S) \leq TV^c\,(Y, S_1) + c + TV^c\,(Y, [S_1; S])$, which follows easily from the estimate

$$(|Y\,(t) - Y\,(u)| - c)_+ \leq (|Y\,(t) - Y\,(S_1)| - c)_+ + (|Y\,(S_1) - Y\,(u)| - c)_+ + c$$

for $0 \leq t < S_1 < u \leq S$, and the Markov property of X we get

$$\mathbf{E}\exp\left(\lambda\left[TV^c\,(Y, S) - c^{-1}SA\right]\right)$$
$$\leq \mathbf{E}\exp\left(\lambda TV^c\,(Y, S_1) + \lambda c + \lambda TV^c\,(Y, [S_1; S_2]) - \lambda c^{-1}SA\right)$$
$$= e^{\lambda c}\mathbf{E}\left(e^{\lambda TV^c(Y, S_1) - \lambda c^{-1}S_1 A}E\left[e^{\lambda TV^c(Y, [S_1; S]) - \lambda c^{-1}S_2 A}|X\,(S_1)\right]\right)$$

$$(4.12)\qquad \leq e^{\lambda c}\frac{1}{1 - \lambda S_1 B/c}\frac{1}{1 - \lambda S_2 B/c}.$$

The last inequality follows by (4.11), since the right hand side of (4.11) does not depend on x_0 and using the Markov property in similar way we have the universal estimate for the conditinal expectation

$$E\left(\exp\left\{\lambda TV^c\,(Y, [S_1; S]) - \lambda c^{-1}S_2 A\right\}|X\,(S_1) = x_1\right) \leq \frac{1}{1 - \lambda S_2 B/c}$$

(note that the length of interval $[S_1; S]$ is S_2).

Notice now that from (4.12) it follows that $\mathbf{E}\exp\left(\lambda\left[TV^c\,(Y, S) - c^{-1}S\bar{A}\right]\right) < +\infty$ for $\lambda < \min\left\{c\,(S_1 B)^{-1}, c\,(S_2 B)^{-1}\right\}$. Let us fix integer $n \geq 1$. Iterating (4.12) we obtain

$$(4.13)\quad \mathbf{E}\exp\left(\lambda\left[TV^c\,(Y, S) - c^{-1}SA\right]\right) \leq e^{\lambda c(n-1)}\left(\frac{1}{1 - \lambda SB\,(cn)^{-1}}\right)^n$$

for $\lambda < cn\,(SB)^{-1}$, which gives that $\mathbf{E}\exp\left(\lambda\left[TV^c\,(Y, S) - c^{-1}SA\right]\right) < +\infty$ for any $\lambda \in \mathbb{R}$. Now, let us fix $\lambda > 0$ and set $n = \left\lceil 2\lambda SBc^{-1}\right\rceil$. Using (4.13) we get

$$\mathbf{E}\exp\left(\lambda\left[TV^c\,(Y, S) - c^{-1}SA\right]\right) \quad\leq\quad e^{\lambda c(n-1)}2^n$$
$$\leq\quad 2\exp\left(2\lambda^2 SB + 2\,(\ln 2)\,\lambda SBc^{-1}\right).$$

and thus

$$\mathbf{E}\exp\left(\lambda TV^c\,(Y, S)\right) \quad\leq\quad 2\exp\left(2\lambda^2 SB + \lambda Sc^{-1}\,(A + 2\,(\ln 2)\,B)\right).$$
$$(4.14)\qquad\qquad\qquad =\quad 2\exp\left(\lambda^2 S\alpha_R + \lambda Sc^{-1}\beta_R\right),$$

where $\alpha_R = 2B = 2B\,(R)$ and $\beta_R = A + 2\,(\ln 2)\,B = A\,(R) + 2\,(\ln 2)\,B\,(R)$.

Now, from (4.7), (4.6) and (4.14) we get

$$\mathbf{E}\exp\left(\lambda TV^c\,(X, S)\right) \quad\leq\quad E\exp\left(\lambda TV\,(M, S) + \lambda TV^c\,(Y, S)\right)$$
$$\leq\quad 2\exp\left(\lambda^2 S\alpha_R + \lambda Sc^{-1}\beta_R + \lambda\gamma_{x_0, C, D, S}\right)E\exp\left(\lambda\delta_{D, S}Y^*\right)$$

where $\gamma_{x_0, C, D, S} = (C + D\,|x_0|)\,Se^{DS}$, $\delta_{D, S} = DSe^{DS}$.

Finally, using (4.10) with $\delta = \lambda\delta_{D,S}$ we get

$$
\begin{aligned}
\mathbf{E}\exp\left(\lambda TV^c\left(X,S\right)\right) \;\leq\; & 2\exp\left(\lambda^2 S\alpha_R + \lambda Sc^{-1}\beta_R + \lambda\gamma_{x_0,C,D,S}\right) \times \\
& \times\left(1 + 8\lambda\eta_{D,R,S}\exp\left(\lambda^2\eta_{D,R,S}^2\right)\right),
\end{aligned}
$$

where $\eta_{D,R,S} = \delta_{D,S}R\sqrt{S/2}$.

$\blacksquare$

Remark 5. *Let us notice that the condition that σ is bounded is essential for obtaining the Gaussian concentration of $TV^c\left(X,S\right)$. To see this it is enough to consider the equation $dX\left(t\right) = 2^{-1}X(t)dt + X\left(t\right)dW\left(t\right)$ with starting condition $X\left(0\right) = 1$. Notice that $TV^c\left(X,S\right) \geq \left(X\left(S\right) - X(0) - c\right)_+$ and that $\left(X\left(S\right) - X(0) - c\right)_+ = \left(\exp W\left(S\right) - 1 - c\right)_+$ does not reveal Gaussian concentration.*

Remark 6. *Notice that for the standard Brownian motion $X = W$, Sc^{-1} is comparable up to a universal constant with $TV^c\left(W,S\right)$. Hence, from (4.3) we obtain that for some universal constants $\bar{A}, \bar{B}$ the Gaussian concentration holds*

$$
\mathbf{P}(TV^c(W,S) \geq \bar{A}\mathbf{E}TV^c(W,S) + \bar{B}\sqrt{S}u) \leq \exp(-u^2), \quad for\ u \geq 0.
$$

5. Existence of moment-generating functions of the truncated variation of Lévy processes

In this section we will deal with the existence of finite exponential moments of the truncated variation of Lévy processes. We will state the necessary and sufficient condition for the finiteness of $\mathbf{E}\exp\left(\alpha TV^c\left(X,S\right)\right)$ in terms of the generating triplet of process X (cf. [14, Chapt. 2, Sect. 11]). The methodology used here is very similar to the methodology used in [10], where the existence of $\mathbf{E}\exp\left(\alpha TV^c\left(W,S\right)\right)$ for Wiener process W and any complex α was proved.

We start with

Lemma 4. *Let X be a Lévy process. For any $c > 0$ and $\alpha > 0$ the finiteness of $\mathbf{E}\exp\left(\alpha TV^c\left(X,S\right)\right)$ is equivalent with the finiteness of*

$$
\mathbf{E}\exp\left(\alpha \sup_{0\leq s\leq S}\left|X\left(s\right)\right|\right).
$$

Proof. ($\Rightarrow$) From the inequality

$$
\begin{aligned}
TV^c\left(X,S\right) &\geq \sup_{0\leq s\leq S}\max\left\{\left|X\left(s\right)\right| - \left|X\left(0\right)\right| - c, 0\right\} \\
&= \max\left\{\sup_{0\leq s\leq S}\left|X\left(s\right)\right| - c, 0\right\} \geq \sup_{0\leq s\leq S}\left|X\left(s\right)\right| - c
\end{aligned}
$$

it follows that for every real α, if $\mathbf{E}\exp\left(\alpha TV^c\left(X,S\right)\right) < +\infty$ then

$$\mathbf{E}\exp\left(\alpha\sup_{0\leq s\leq S}|X\left(s\right)|\right) = e^c\mathbf{E}\exp\left(\alpha\left\{\sup_{0\leq s\leq S}|X\left(s\right)| - c\right\}\right)$$
$$\leq e^c\mathbf{E}\exp\left(\alpha TV^c\left(X,S\right)\right) < +\infty.$$

($\Longleftarrow$) To prove the opposite implication let us define $T_0^c = 0$ and for $i = 1, 2, \ldots$

$$T_i^c = \begin{cases} \inf\left\{t > T_{i-1}^c : \left|X\left(t\right) - X\left(T_{i-1}^c\right)\right| > c/2\right\} \wedge \left(S + T_{i-1}^c\right) & \text{if } T_{i-1}^c < +\infty; \\ +\infty & \text{otherwise.} \end{cases}$$

Observe that $T_1^c = \inf\left\{t > 0 : |X\left(t\right)| > c/2\right\} \wedge S \leq S$ and that $\left(X\left(t\right)\right)_{t\geq 0} =^d$ $\left(X\left(t\right) - X\left(T_1^c\right)\right)_{t\geq T_1^c}$, where "$=^d$" denotes equality of distributions. Now let us define

$$X_t^c = \sum_{i=0}^{\infty} X\left(T_i^c\right) I_{\left[T_i^c, T_{i+1}^c\right)}\left(t\right).$$

Since $\|X^c - X\|_\infty \leq c/2$, we have

(5.1) $$TV^c\left(X,S\right) \leq TV\left(X^c,S\right)$$

and since X^c is picewise constant with the first jump at $T_1^c \leq S$, we have

(5.2) $$TV\left(X^c,S\right) = |\Delta X^c\left(T_1^c\right)| + TV\left(X^c,[T_1^c;S]\right)$$
$$\leq \sup_{0\leq s\leq T_1^c}|X\left(s\right)| + TV\left(X^c,[T_1^c;S]\right).$$

Let now $\delta \in (0;S)$ be such a small number that
(5.3)
$$\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq S}|X\left(s\right)|\right); T_1^c \leq \delta\right] := \mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq S}|X\left(s\right)|\right) I_{\left\{T_1^c \leq \delta\right\}}\right] < 1$$

(Note that such a number exists, since we assume that $\mathbf{E}\exp\left(\alpha\sup_{0\leq s\leq S}|X\left(s\right)|\right) < +\infty$ and from the càdlàg property and stochastic continuity of X it follows that $\mathbf{P}\left(T_1^c \leq \delta\right) = \mathbf{P}\left(\sup_{0\leq s\leq\delta}|X\left(s\right)| > c/2\right) \downarrow 0$ as $\delta \downarrow 0$.)

Fix $M > 0$. Applying (5.2), the independence of the process $X\left(t\right) - X\left(T_1^c\right), t \geq T_1^c$, and the two-dimensional r.v. $\left(\sup_{0\leq s\leq T_1^c}|X\left(s\right)|, T_1^c\right)$ (strong Markov property of Lévy processes) and the equality of distributions of

$TV\left(X^{c},s\right)$ and $TV\left(X^{c},[T_{1}^{c};T_{1}^{c}+s]\right)$ for any $s\geq 0$, we have

$$\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)=\mathbf{E}\left[\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right);T_{1}^{c}\leq\delta\right]+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$\leq\mathbf{E}\left[\exp\left(\alpha\left(\sup_{0\leq s\leq T_{1}^{c}}\left|X\left(s\right)\right|+TV\left(X^{c};[T_{1}^{c};\delta]\right)\right)\wedge M\right);T_{1}^{c}\leq\delta\right]+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$\leq\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq T_{1}^{c}}\left|X\left(s\right)\right|+\alpha TV\left(X^{c};[T_{1}^{c};\delta]\right)\wedge M\right);T_{1}^{c}\leq\delta\right]+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$\leq\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq T_{1}^{c}}\left|X\left(s\right)\right|+\alpha TV\left(X^{c};[T_{1}^{c};\delta+T_{1}^{c}]\right)\wedge M\right);T_{1}^{c}\leq\delta\right]+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$=\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq T_{1}^{c}}\left|X\left(s\right)\right|\right);T_{1}^{c}\leq\delta\right]\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$\leq\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq\delta}\left|X\left(s\right)\right|\right);T_{1}^{c}\leq\delta\right]\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

$$\leq\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq S}\left|X\left(s\right)\right|\right);T_{1}^{c}\leq\delta\right]\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)+\mathbf{P}\left(T_{1}^{c}>\delta\right).$$

Thus we have obtained that

$$\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)\leq\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq S}\left|X\left(s\right)\right|\right);T_{1}^{c}\leq\delta\right]\times$$

$$\times\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)+\mathbf{P}\left(T_{1}^{c}>\delta\right)$$

and by (5.3) we have
(5.4)
$$\mathbf{E}\exp\left(\alpha TV\left(X^{c},\delta\right)\wedge M\right)\leq\frac{\mathbf{P}\left(T_{1}^{c}>\delta\right)}{1-\mathbf{E}\left[\exp\left(\alpha\sup_{0\leq s\leq S}\left|X\left(s\right)\right|\right);T_{1}^{c}\leq\delta\right]}.$$

By similar arguments as before (i.e. (5.2), independence of $X\left(t\right)-X\left(T_{1}^{c}\right),t\geq T_{1}^{c}$, and $\left(\sup_{0\leq s\leq T_{1}^{c}}\left|X\left(s\right)\right|,T_{1}^{c}\right)$ and the equality of distributions of $TV\left(X^{c},s\right)$

and $TV\left(X^c, [T_1^c; T_1^c + s]\right)$ for $s \geq 0$) we have

$$\mathbf{E}\exp\left(\alpha TV\left(X^c, S\right) \wedge M\right)$$

$$\leq \mathbf{E}\exp\left(\left(\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S]\right)\right) \wedge M\right)\right)$$

$$\leq \mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S]\right) \wedge M\right)$$

$$= \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S]\right) \wedge M\right); T_1^c \leq \delta\right]$$

$$+ \mathbf{E}\exp\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S]\right) \wedge M\right); T_1^c > \delta\right]$$

$$\leq \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S + T_1^c]\right) \wedge M\right); T_1^c \leq \delta\right]$$

$$+ \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)| + \alpha TV\left(X^c; [T_1^c; S + T_1^c - \delta]\right) \wedge M\right); T_1^c > \delta\right]$$

$$= \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)|\right); T_1^c \leq \delta\right]\mathbf{E}\left[\exp\left(\alpha TV\left(X^c, S\right) \wedge M\right)\right]$$

$$+ \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq T_1^c}|X(s)|\right); T_1^c > \delta\right]\mathbf{E}\left[\exp\left(\alpha TV\left(X^c, S - \delta\right) \wedge M\right)\right]$$

$$\leq \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq S}|X(s)|\right); T_1^c \leq \delta\right]\mathbf{E}\left[\exp\left(\alpha TV\left(X^c, S\right) \wedge M\right)\right]$$

$$+ \mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S}|X(s)|\right)\mathbf{E}\left[\exp\left(\alpha TV\left(X^c, S - \delta\right) \wedge M\right)\right].$$

From this we have

$$\mathbf{E}\exp\left(\alpha TV\left(X^c, S\right) \wedge M\right)$$

$$\leq \frac{\mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S}|X(s)|\right)}{1 - \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq S}|X(s)|\right); T_1^c \leq \delta\right]}\mathbf{E}\exp\left(\alpha TV\left(X^c, S - \delta\right) \wedge M\right).$$

Similarly, if $S - 2\delta > 0$

$$\mathbf{E}\exp\left(\alpha TV\left(X^c, S - \delta\right) \wedge M\right)$$

$$\leq \frac{\mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S - \delta} |X(s)|\right)}{1 - \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq S - \delta} |X(s)|\right); T_1^c \leq \delta\right]} \mathbf{E}\exp\left(\alpha TV\left(X^c, S - 2\delta\right) \wedge M\right)$$

$$\leq \frac{\mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right)}{1 - \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right); T_1^c \leq \delta\right]} \mathbf{E}\exp\left(\alpha TV\left(X^c, S - 2\delta\right) \wedge M\right).$$

Iterating and putting together the above inequalities we finally obtain

$$\mathbf{E}\exp\left(\alpha TV\left(X^c, S\right) \wedge M\right) \leq \left(\frac{\mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right)}{1 - \mathbf{E}\left[\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right); T_1^c \leq \delta\right]}\right)^{\lfloor S/\delta \rfloor} \times$$

(5.5) $$\times \mathbf{E}\exp\left(\alpha TV\left(X^c, \delta\right) \wedge M\right).$$

By (5.4) and (5.5), and letting $M \to \infty$ we get $\mathbf{E}\exp\left(\alpha TV\left(X^c, S\right)\right) < +\infty$. Finally, from (5.1) we get

$$\mathbf{E}\exp\left(\alpha TV^c\left(X, S\right)\right) < +\infty.$$

∎

Now let (A, ν, γ) be the generating triplet of the process X. By [14, Theorem 28.15] we have

$$\mathbf{E}\exp\left(\alpha \sup_{0 \leq s \leq S} |X(s)|\right) < +\infty$$

if and only if

$$\mathbf{E}\exp\left(\alpha |X(1)|\right) < +\infty$$

which, by [14, Corollary 25.8], is equivalent with

$$\int_{|x| > 1} e^{\alpha |x|} \nu(dx) < +\infty.$$

Thus we have obtained

Theorem 3. *Let (A, ν, γ) be the generating triplet of the Lévy process X. For any $\alpha > 0$ we have*

$$\mathbf{E}\exp\left(\alpha TV^c\left(X, S\right)\right) < +\infty$$

if and only if

$$\int_{|x| > 1} e^{\alpha |x|} \nu(dx) < +\infty.$$

References

[1] Bednorz, W. A theorem on majorizing measures, *Annals of Probability*, **34**, no. 5, 1771-1781, (2006).

[2] Bednorz, W. On a Sobolev type inequality and its applications *Studia Mathematica*, **176**, no. 2, 113-137, (2006)

[3] Bednorz, W. (2010), Majorizing measures on metric spaces, *C.R. math. Acad. Sci. Paris*, **348**, no. 1-2, 75-78, (2010)

[4] Li, W.V. and Shao Q.M. (2001) Gaussian Processes, Small Ball Probabilities and Applications. *Stoch. Proc.: Theory and methods* Handbook of Statistics, Elsevier.

[5] Fernique, X. Caractérisation de processus á trajectoires majorées ou continues. (French) *Séminaire de Probabilités XII.* Lecture Notes in Math. **649**, 691–706, Springer, Berlin. (1978)

[6] Fernique, X. Régularité de fonctions aléatoires non gaussiennes. (French) *École d'Été de Probabilités de Saint-Flour XI-1981.* Lecture Notes in Math. **976**, 1–74, Springer, Berlin. (1983)

[7] Kwapień, S. and Rosiński, J. Sample Hölder continuity of stochastic processes and majorizing measures. *Seminar on Stochastic Analysis, Random Fields and Applications IV, Progr. in Probab.* **58**, 155–163. Birkhäuser, Basel. (2004)

[8] Ledoux, M. and Talagrand, M. Probability in Banach spaces. Isoperimetry and processes. *Results in Math. and Rel. Areas (3).* **23**, xii+480 pp. Springer-Verlag, Berlin. (1991)

[9] Łochowski, R. On the Generalisation of the Hahn-Jordan Decomposition for Real Càdlàg Functions. *arXiv:1210.3932*

[10] Łochowski, R. Truncated variation, upward truncated variation and downward truncated variation of Brownian motion with drift – Their characteristics and applications. *Stochastic Proc. Appl.* **121**, no. 2, 378-393 (2011)

[11] Łochowski, R. On pathwise uniform approximation of processes with càdlàg trajectories by processes with finite total variation. *arXiv:1106.2630*

[12] Łochowski, R., Miłoś, P. On truncated variation, upward truncated variation and downward truncated variation for diffusions. *Stochastic Proc. Appl.* **123**, no. 2, 446-474 (2013)

[13] Revuz, D. and Yor, M. Continuous martingales and Brownian motion, volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, (2005)

[14] Sato, K.-I., Lévy Processes and Infinite Divisibility. *Cambridge Univ. Press.* (1999)

[15] Talagrand, M. Sample boundedness of stochastic processes under increment conditions. *Annals of Probability* **18**, no. 1, 1-49 (1990)

[16] Talagrand, M. The generic chaining. *Springer-Verlag.* (2005)

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